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On strongly regular graphs with $\mu \leqslant 2$

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Abstract

In this paper we prove that any strongly regular graph with $\mu = 1$ satisfies $k \geqslant (\lambda+1)(\lambda+2)$ and any strongly regular graph with $\mu = 2$ is either a grid graph or satisfies $k \geqslant \frac{1}{2}\lambda(\lambda+3)$. This improves upon a previous result of Brouwer and Neumaier who gave a necessary restriction on the parameters of strongly regular graphs with $\mu = 2$ and $k < \frac{1}{2}\lambda(\lambda+3)$.

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Such a graph G is said to be strongly regular (cf. [3]) with parameters v, k, λ, μ if it satisfies: (a) G has v vertices, (b) each vertex of G has exactly k neighbours, (c) any two adjacent vertices of G have exactly λ common neighbours, and (d) any two (distinct) non-adjacent vertices of G have exactly μ common neighbours. Clearly a disconnected strongly regular graph is a disjoint union of complete graphs of equal size. In this note we exclude these trivial examples and their complements. In other words, all strongly regular graphs considered here are primitive (i.e., connected and co-connected). That is, we assume $0 < \mu < k < v - 1$.

Clearly, the complement of a strongly regular graph with parameters (v, k, λ, μ) is a strongly regular graph with parameters $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$. In consequence, we have the trivial necessary condition $v - 2k + \mu - 2 \geqslant 0$ for the existence of a strongly regular graph. A simple counting argument shows that we also have the necessary condition $k(k-\lambda-1) = (v-k-1)\mu$ (but this is a consequence of the integrality condition mentioned below).

It is easy to see that the zero–one adjacency matrix of a (primitive) strongly regular graph has only three distinct eigenvalues, of which k is a simple eigenvalue. The remaining two eigenvalues are usually denoted by $r > s$. Further, the corresponding multiplicities are denoted by f and g. In the so called ‘half-case’ $f = g$, the parameters are $(4\mu+1, 2\mu, \mu-1, \mu)$ and the eigenvalues are $r = \frac{1}{2}(-1+\sqrt{v})$ and $s = \frac{1}{2}(-1-\sqrt{v})$. In all the remaining cases, r and s are integers. The parameters are given in terms of the eigenvalues by the formulae:

$$v = \frac{(k-r)(k-s)}{(k+rs)}, \quad k = k, \quad \lambda = k + rs + r + s, \quad \mu = k + rs.$$

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Further, the multiplicities are given by:

$$f = \frac{-k(s+1)(k-s)}{(k+rs)(r-s)}, \quad g = \frac{k(r+1)(k-r)}{(k+rs)(r-s)}.$$

Since these multiplicities must be integers, we have further restrictions on the parameters of a strongly regular graph. Any quadruple (v, k, λ, μ) satisfying the restrictions mentioned so far is said to be feasible for a strongly regular graph. Many more non-trivial restrictions on the parameters of a strongly regular graph are known. These necessary conditions are surveyed in [4]. Though this survey is now 14 years old, as far as we know no further restrictions of a general nature have been found after its publication. The known necessary conditions include a result of Bose and Dowling from [2]: when $\mu = 1$, $\lambda + 1$ divides k and $(\lambda + 1)(\lambda + 2)$ divides vk.

The objective of this paper is to show that an argument similar to the one in [2], in conjunction with Hoffman's bound and a result of Brouwer and Neumaier from [5], yields the restriction on the parameters of strongly regular graphs mentioned in the abstract. Recall that a clique in a graph is a set of pair-wise adjacent vertices. A clique in a regular graph is said to be regular if there is a constant α (called the nexus of the clique) such that every vertex outside the clique is adjacent to exactly α vertices inside the clique. Hoffman's bound (cf. [3, p. 10]) says (in particular) that any clique in a strongly regular graph has size $\leqslant (k/-s) + 1$, with equality iff the clique is regular. Further, in the case of equality, the nexus of the clique equals $\mu/-s$.

We also recall that a partial linear space is an incidence system with at most one line through any two distinct points. The collinearity graph of a partial linear space is the graph with the points as vertices, where two distinct vertices are adjacent iff the corresponding points are joined by a line. Note that each line is a clique in the collinearity graph. A partial geometry with parameters (s, t, α) is a partial linear space with $s+1$ points on each line and $t+1$ lines through each point, such that for any line l and any point x outside l, exactly α points on l are collinear with x. Thus, each line of a partial geometry is a regular clique in its collinearity graph, of size $s+1$ and nexus α. A generalized quadrangle is a partial geometry of nexus $\alpha = 1$. By [1], the collinearity graph of a partial geometry with parameters (s, t, α) is a strongly regular graph with parameters

$$v = (s+1)\left(\frac{st}{\alpha} + 1\right), \quad k = s(t+1), \quad \lambda = s - 1 + t(\alpha - 1), \quad \mu = (t+1)\alpha.$$

In particular, the collinearity graph of an (s, t)-generalized quadrangle is a strongly regular graph with parameters

$$v = (s+1)(st+1), \quad k = s(t+1), \quad \lambda = s-1, \quad \mu = t+1.$$

Finally, given two graphs G and H, we say that G is H-free if G has no induced subgraph isomorphic to H. As usual, we denote the complete tripartite graph on $1+1+2$ vertices by $K_{1,1,2}$. In other words, $K_{1,1,2}$ is the complete graph on four vertices minus an edge. Note that the collinearity graph of a generalized quadrangle is $K_{1,1,2}$-free. We are now ready to state:

Lemma 1. *Any $K_{1,1,2}$-free strongly regular graph is either the collinearity graph of a generalized quadrangle, or else its parameters satisfy $k \geqslant (\lambda+1)(\lambda+2)$.*

Proof. Since the graph is $K_{1,1,2}$-free, any two adjacent vertices together with their common neighbours form a clique of size $\lambda + 2$. Clearly any two adjacent vertices are in a unique clique in this family, and each vertex is in $k/(\lambda+1)$ of them. In other words, these cliques are the lines of a partial linear space with $k/(\lambda+1)$ lines through each point and $\lambda+2$ points on each line; the given graph is its collinearity graph. The total number of lines is $b = vk/(\lambda+1)(\lambda+2)$. Let N be the $v \times b$ incidence matrix of this partial linear space. We have $NN^T = k/(\lambda+1)I + A$, where A is the adjacency matrix of the graph. If $k < (\lambda+1)(\lambda+2)$ then $b < v$ and hence NN^T is singular. Since NN^T is non-negative definite, it follows that its minimum eigenvalue equals 0. That is, the minimum eigenvalue of A is $s = -k/(\lambda+1)$. Thus, the lines are cliques of size $\lambda + 2 = (k/-s) + 1$, which is the Hoffman bound. Because of equality in Hoffman's bound, these cliques are regular. Since no two vertices on a line have a common neighbour outside a line, it is a clique of nexus =1. Thus, the partial linear space is a generalized quadrangle with the given graph as its collinearity graph. $\square$

Corollary 2. *The parameters of a strongly regular graph with* $\mu = 1$ *must satisfy* $k \geqslant (\lambda + 1)(\lambda + 2)$. *Equality holds only for the pentagon*.

Proof. Clearly a strongly regular graph with $\mu = 1$ is $K_{1,1,2}$-free and it cannot be the collinearity graph of a generalized quadrangle. Therefore, the inequality is immediate from Lemma 1. This inequality may be rewritten as $-(r+1)/(s+1) \leqslant \frac{1}{2}(3 + \sqrt{5})$. Thus, equality can hold only in the half cases. An inspection shows that among the half case parameters, only that of the pentagon satisfies equality. □

Example 3. This Corollary kills the feasible parameters (1666, 45, 8, 1) and (2745, 56, 7, 1) which survive all the tests mentioned in [4].

Notice that for each positive integer n, there is only one generalized quadrangle with n points on each line and two lines through each point. Its lines are the rows and columns of an $n \times n$ array. The collinearity graph of this geometry is called the $n \times n$ grid graph. In other words, the $n \times n$ grid graph (also called $L_2(n)$ in the literature) is the line graph of the complete bipartite graph $K_{n,n}$. It is strongly regular with parameters $(n^2, 2n - 2, n - 2, 2)$.

Theorem 4. *Any strongly regular graph with* $\mu = 2$ *is either a grid graph or else its parameters satisfy* $k \geqslant \frac{1}{2}\lambda(\lambda + 3)$.

Proof. Suppose $k < \frac{1}{2}\lambda(\lambda+3)$. By a result of [5], any strongly regular graph with $\mu=2$ and $k < \frac{1}{2}\lambda(\lambda+3)$ is $K_{1,1,2}$-free. Therefore, by Lemma 1, the given graph is the collinearity graph of an (s, t)-generalized quadrangle. Since $t + 1 = \mu = 2$, it is a grid graph. □

Remark 5. In [5], the above mentioned result was used to conclude that the parameters of a strongly regular graph with $\mu = 2$ and $k < \frac{1}{2}\lambda(\lambda + 3)$ must satisfy $\lambda + 1 | k$ and $(\lambda + 1)(\lambda + 2) | vk$. Now, Theorem 4 shows that this conclusion is almost vacuous: only the $n \times n$ grids (with $n \geqslant 5$) satisfy the hypothesis.

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